The non-thermal universe
Part III

Nonlinear theory of DSA, field amplification and relativistic shocks

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Reminder and summary

- Shocks are self-forming 1-D dynamical structures which convert mechanical flow energy into random particle motion.
- Ubiquitous in SNRs, stellar wind bubbles, accretion flows etc...
- Scale of shock determined by the physics of the dissipation - molecular mean free path for gas-dynamic shocks, thermal ion gyroradius or other relevant plasma scale

- Astrophysical plasmas are very low density and collisionless - they behave as fluids only because of the long-range collective coupling through the electromagnetic field.

- Magnetic fields can easily drive plasma velocity distribution to near isotropy, but not to thermal equilibrium.

- Non-thermal distributions can survive for very long times.

- Diffusive Shock Acceleration is a variant of Fermi acceleration that operates at collisionless shocks

- Produces non-thermal power-law tails extending to high energy on the downstream particle momentum distribution function.

Summary of test-particle (linear) theory

- Shock is simple jump discontinuity

$$U(x) = \begin{cases} U_1, & x < 0, \\ U_2, & x > 0 \end{cases}$$

- Particles have power-law spectrum in momentum

$$f(p) \propto p^{-s}, \qquad s = \frac{3U_1}{U_1 - U_2}$$

- Acceleration time-scale is

$$t_{acc} = \frac{3}{U_1 - U_2}\left(\frac{\kappa_1}{U_1} + \frac{\kappa_2}{U_2}\right)$$

But easy to show that accelerated particle pressure can be significant, so must worry about reaction effects. Also, if process is to work with high efficiency, as appears to be required, eg, to explain the Galactic cosmic ray origin, we need a nonlinear theory.

In principle easy - we just have to solve the diffusive transport equation and the usual hydrodynamic equations with an additional cosmic ray pressure in the momentum equation!

$$P_C(x) = \int \frac{4\pi p^3 v}{3} f(p,x)\, dp$$

In practice very hard!

Possible approaches

- Throw it at the computer
- Monte-Carlo approach
- Two-fluid approximation
- Semi-analytic theories

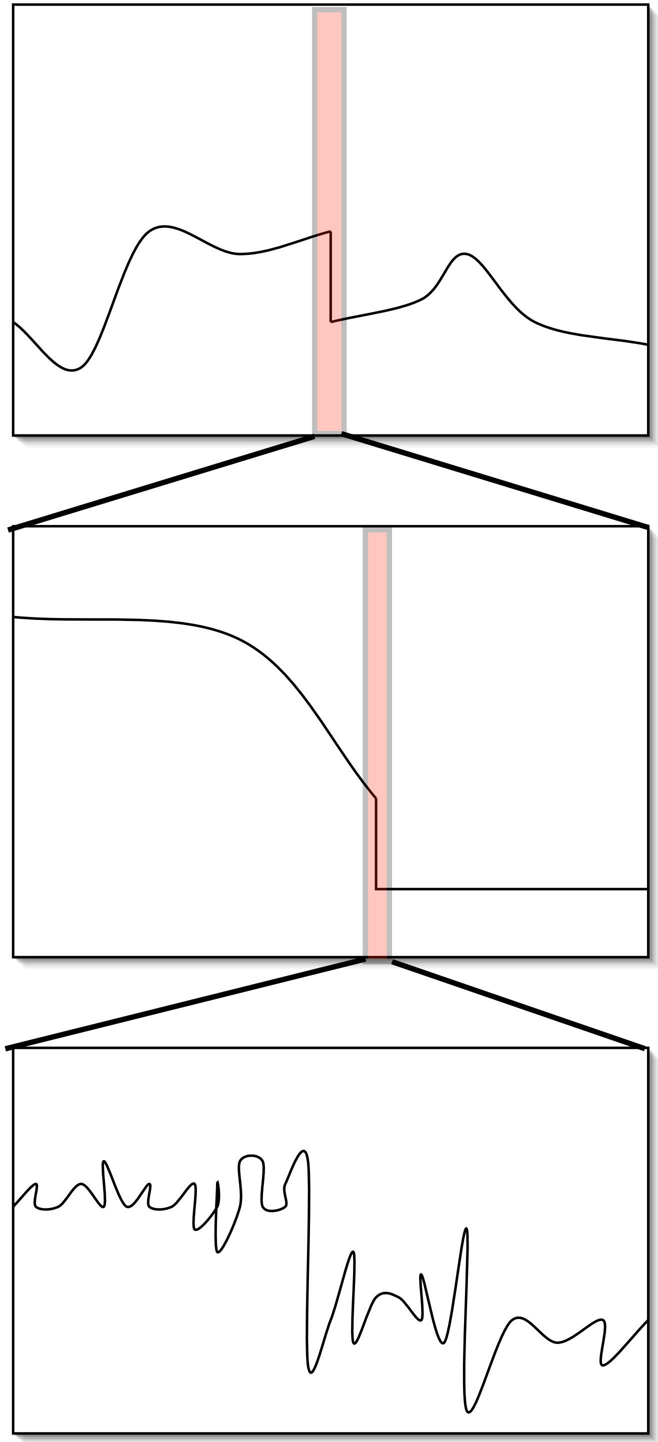
Good general agreement now
between all approaches!

Very wide scale separation - numerical nightmare, but useful for analytic approaches. Can distinguish two extreme scales..

Outer scale of macroscopic system and maximum energies

Inner scale of injection processes and kinetic effects

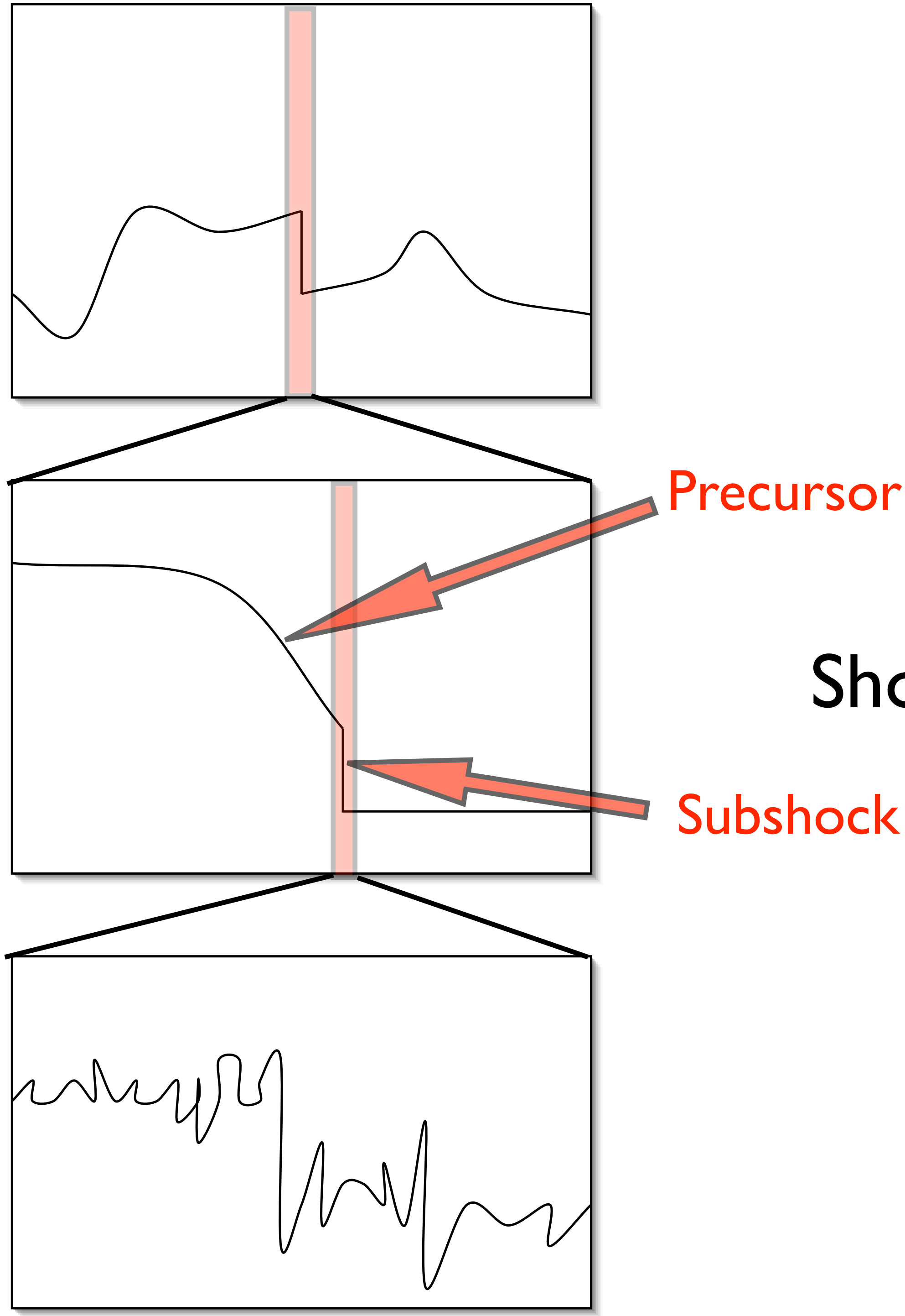
Aim of analytic theory should be to bridge the gap between these two regimes (mesoscopic theory), but not to try to be a complete theory. Analogy to inertial range theories of turbulence.



Outer scale
Astrophysics

Intermediate scales
Shock acceleration theory

Inner scale
Plasma physics



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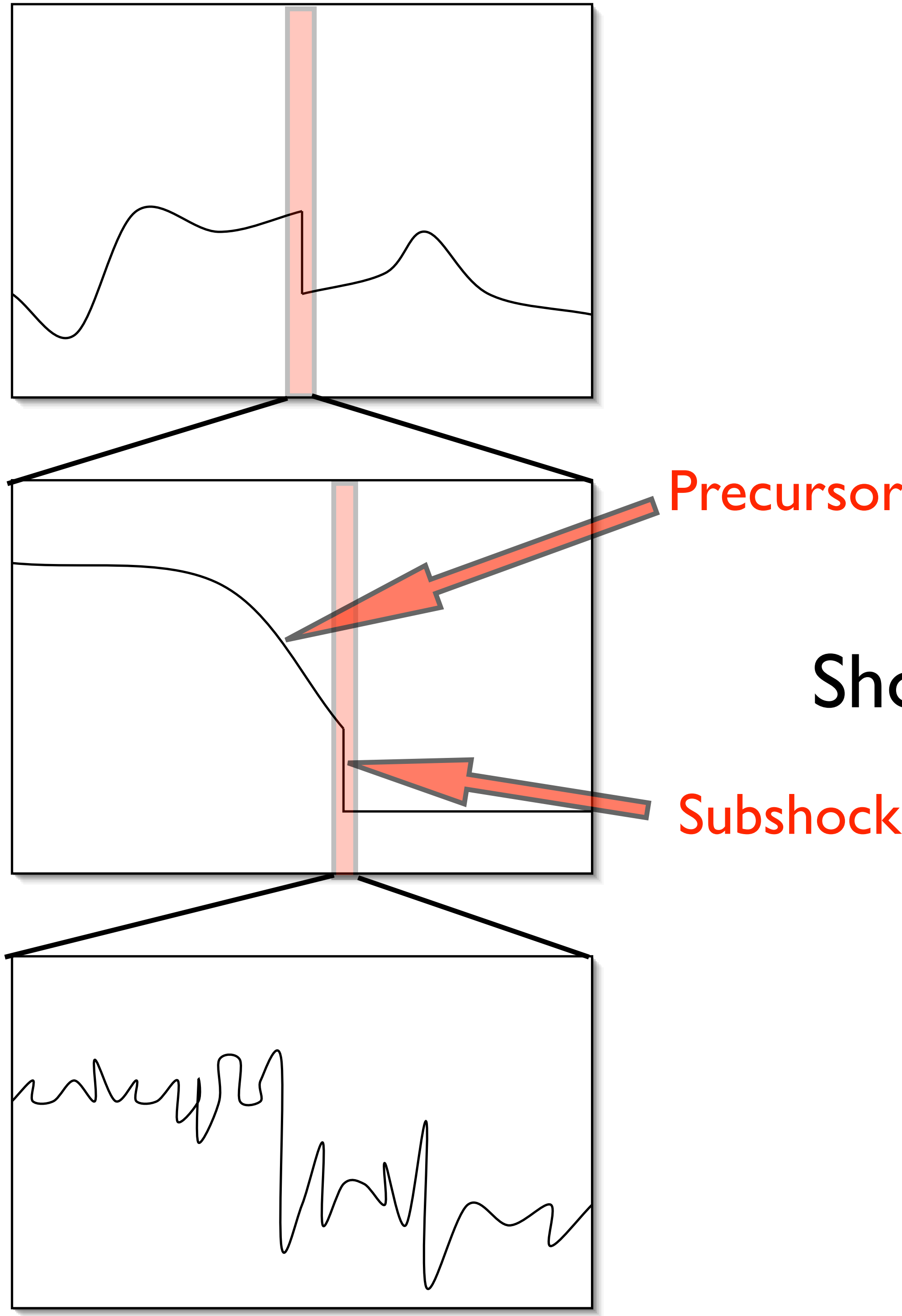

Outer scale
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Intermediate scales
Shock acceleration theory

Inner scale
Plasma physics
Injection!

Shock modification

- Extended upstream precursor + subshock structure
- Increased total compression due to
  - softer equation of state
  - additional energy flux to high energy particles (escape, geometrical dilution, diffusion)

Aside on compression in a strong shock....

$$
\begin{aligned}
\rho_1 U_1 &= \rho_2 U_2 = A \\
A U_1 &= A U_2 + P_2 \\
\frac{1}{2} A U_1^2 &= \frac{1}{2} A U_2^2 + U_2(\mathcal{E}_2 + P_2) + \Phi
\end{aligned}
$$

$$
\Longrightarrow \frac{U_1}{U_2} = 1 + \frac{2\mathcal{E}_2}{P_2} + \frac{2\Phi}{U_2 P_2}
$$

Typically see compression rations of 10 and more in simulations

- Spectrum at low energies given by test-particle theory applied to the sub-shock, thus softer.

- Spectrum at high energies should reflect much increased compression of total shock structure, thus harder

- Concave spectrum - no longer perfect power-law.

Semi-analytic approach to steady mesoscopic structure

Can (hopefully) assume steady planar structure with fixed mass and momentum fluxes.

$$\begin{aligned} \rho U &= A \\ AU + P_G + P_C &= B \end{aligned}$$

and we still have the steady balance between acceleration and loss downstream...

$$\frac{\partial \Phi}{\partial p} = -4\pi p^2 f_0(p) U_2$$

.. but the problem is that the acceleration flux now depends on the upstream velocity profile ***and*** the particle distribution.

However, if one makes an *Ansatz*

$$f_0(p) \to f(x,p)$$

the particle conservation equation and the momentum balance equations, become two coupled equations (in general integro-differential) for the two unknown functions.

$$U(x), \quad f_0(p)$$

An obvious *Ansatz* would be to assume a distribution similar to that familiar from the test-particle theory,

$$f(x,p) = f_0(p) \exp \int \frac{U(x)\,dx}{\kappa(x,p)}$$

This is actually close to Malkov's *Ansatz* who, however, uses

$$f(x,p) = f_0(p) \exp \int \left(-\frac{1}{3} \frac{\partial \ln f_0}{\partial \ln p} \right) \frac{U(x)\,dx}{\kappa(x,p)}$$

which is better for strongly modified shocks.

Motivation comes from exact solution for uniformly distributed compression, *ie* linear velocity field. Easy to check that

$$f = p^{-7/2} \exp\left(\frac{-\alpha x^2}{2\kappa}\right)$$

identically satisfies

$$\frac{\partial f}{\partial t} + U\frac{\partial f}{\partial x} - \frac{1}{3}\frac{\partial U}{\partial x}p\frac{\partial f}{\partial p} = \frac{\partial}{\partial x}\left(\kappa\frac{\partial f}{\partial x}\right)$$

for $\quad U(x) = -x, \qquad \kappa \propto p, \qquad \alpha = \dfrac{7}{6}$

$$
\begin{aligned}
f(x,p) &= p^{-7/2} \exp\left(\frac{-\alpha x^2}{2\kappa}\right) \\
&= f_0(p) \int \left(-\frac{1}{3}\frac{\partial \ln f_o}{\partial \ln p}\right) \frac{U(x)\, dx}{\kappa}
\end{aligned}
$$

So the additional factor introduced by Malkov in the exponential can be thought of as compensating for the fact that the acceleration is distributed over the whole precursor and is not just concentrated at one point.

Blasi introduces a further factor to interpolate between these and recommends the following modified version of Malkov's *Ansatz*

$$f(x,p) = f_0(p) \exp \int \left(-\frac{1}{3} \frac{\partial \ln f_0}{\partial \ln p} \right) \left(1 - \frac{1}{r_{\mathrm{tot}}} \right) \frac{U(x)\, dx}{\kappa(x,p)}$$

Note that all of these are approximations and not exact solutions despite the impressions sometimes given. The good news is that they all give very similar answers....

Remarkably, the crudest *Ansatz*, which simply assumes the accelerated particles penetrate a fixed distance upstream and then abruptly stop, appears to work quite well and gives results very similar to the more complicated ones. This approximation, originally due to Eichler, is

$$f(x,p) = \begin{cases} f_0(p), & x > -L(p) \\ 0, & x < -L(p) \end{cases}$$

It leads to equations which can be heuristically derived in a nonlinear box model and which have been used by a number of authors, most recently P. Blasi and co-workers (their method A).

Defining $U_p = U(-L(p))$

$$\Phi(p) = \int \frac{4\pi p^3}{3} f(x,p) \frac{du}{dx}\, dx$$

$$= \frac{4\pi p^3}{3} f_0 (U_p - U_2)$$

and particle number conservation reads..

$$\frac{d\Phi}{dp} = -4\pi p^2 f_0(p) U_2$$

$$= -\frac{3U_2}{U_p - U_2} \frac{\Phi}{p}$$

Ignoring for the moment the gas pressure momentum balance gives

$$
\begin{aligned}
A dU_p &= \frac{4\pi}{3} p^3 v f_0(p) dp \\
&= \frac{\Phi}{U_p - U_2} v dp
\end{aligned}
$$

So in this simple case get a non-linear box model described by two coupled ODEs.

In general coupled integro-differential equations.

Remarkably, if we ignore gas pressure and switch to particle kinetic energy, rather than momentum, as independent variable, the last equation can be written

$$\frac{\partial}{\partial T}\left(U_p - U_2\right)^2 = \frac{2\Phi}{A}$$

Thus if losses can be neglected and the acceleration flux is a constant

$$U_2 \approx 0, \quad U_p \approx \sqrt{\frac{2\Phi T}{A}}, \quad f_0 \propto p^{-3} T^{-1/2}$$

which is just Malkov's "universal" spectrum

Can be thought of as the asymptotic attractor for all nonlinearly modified solutions at high energies.

Power-law spectrum hardens to 3.5

No particle escape!

Precursor velocity profile is linear.

Corresponds to accelerator going flat-out, all energy goes into flux of particles upwards in energy space...

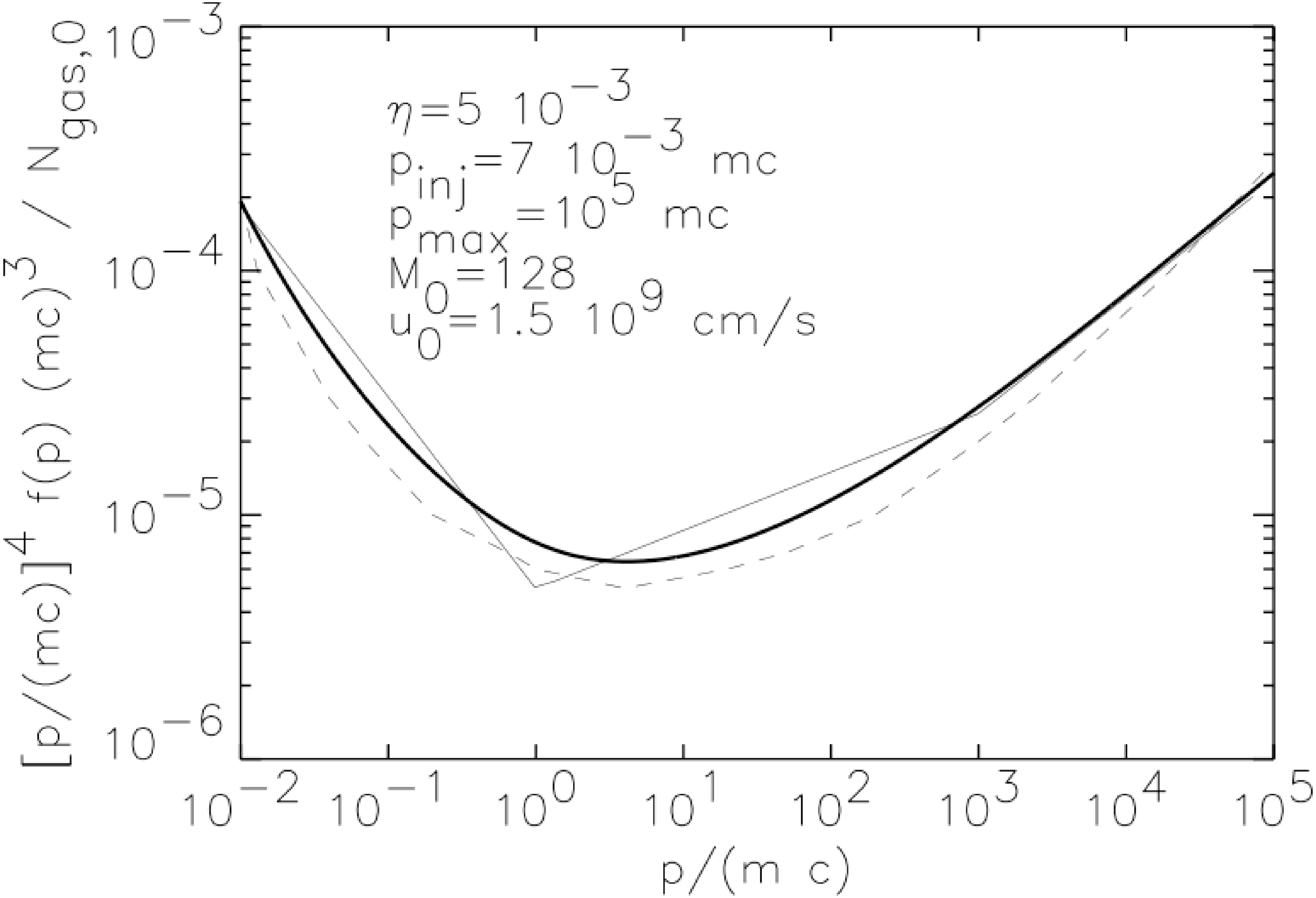
Reality lies in between. If the shock modifies itself to the point that the injection of ions is reduced to the level required, then the subshock has to be weakened, but still exist. Because the pressure per logarithmic momentum interval is

$$\frac{4\pi}{3} p^4 v f(p)$$

in the asymptotic limit of a very low injection energy, the subshock has to have a compression ratio of at least 2.5, corresponding to a local spectral index of 5,

$$\frac{3 \times 2.5}{2.5 - 1} = 5$$

If the pressure is not to fall away too rapidly.



From P. Blasi, 2002

Has important consequences for the post-shock gas temperature. If indeed the shock is self-regulated by the need to reduce the sub-shock compression to a value of order 2.5, then the post-shock gas temperature is fixed not so much by the shock speed as by the upstream temperature.

The postshock temperature is a result of adiabatic compression in the precursor followed by shock heating in the subshock, and if the total compression is, say, 10 with 4 in the precursor and 2.5 in the subshock, then the gas temperature rises by a factor of $4^{\gamma-1}$ in the precursor and a further factor of 12/5 in the subshock - in total a modest factor 6.

Consensus view...

- Spectra are generically curved, softer at low energies, hardening in the relativistic region before cutting off.
- Hardening at high energies at most changes spectral index from 4 to 3.5, so not too extreme
- Subshock is reduced to point where injection matches capacity of shock to accelerate; suggests minimum subshock compression ratio of about 2.5.

But...

- All approaches assume steady structure on the mesoscopic scale.
- In fact exist many possible instabilities.
- However can hope that theory still applies in mean sense - basic physics is very robust.
- Also not all bad news - offers exciting prospect of amplified B fields and thereby reaching higher energies.

The Instability Zoo

- Streaming excitation of Alfven waves (eg Wentzel, 1974; Skilling 1975)
- Acoustic instability (Drury and Falle, 1986)
- Parker instability (1966, 1967)
- McKenzie and Voelk, 1981 - wave heating or "plastic deformation of field".
- Bell and Lucek, 2000, 2001; Bell 2004, 2005
- Generic Weibel-type instabilities
- Jokipii - downstream vorticity

- Streaming excitation of Alfven waves
  - Originally proposed to give enhanced scattering at the shock (Bell 1978)
  - resonant v non-resonant terms (Achterberg, 1983; Bell, 2004; Reville 2006)
  - physical ordering of terms inappropriate for shock precursor case as noted by Bell.

- Acoustic instability
  - Exists in purely 1-D models (and thus important for numerical codes).
  - Depends on collective nature of scattering

For small scale perturbations in 1-D

$$\kappa \nabla P_C \approx const$$

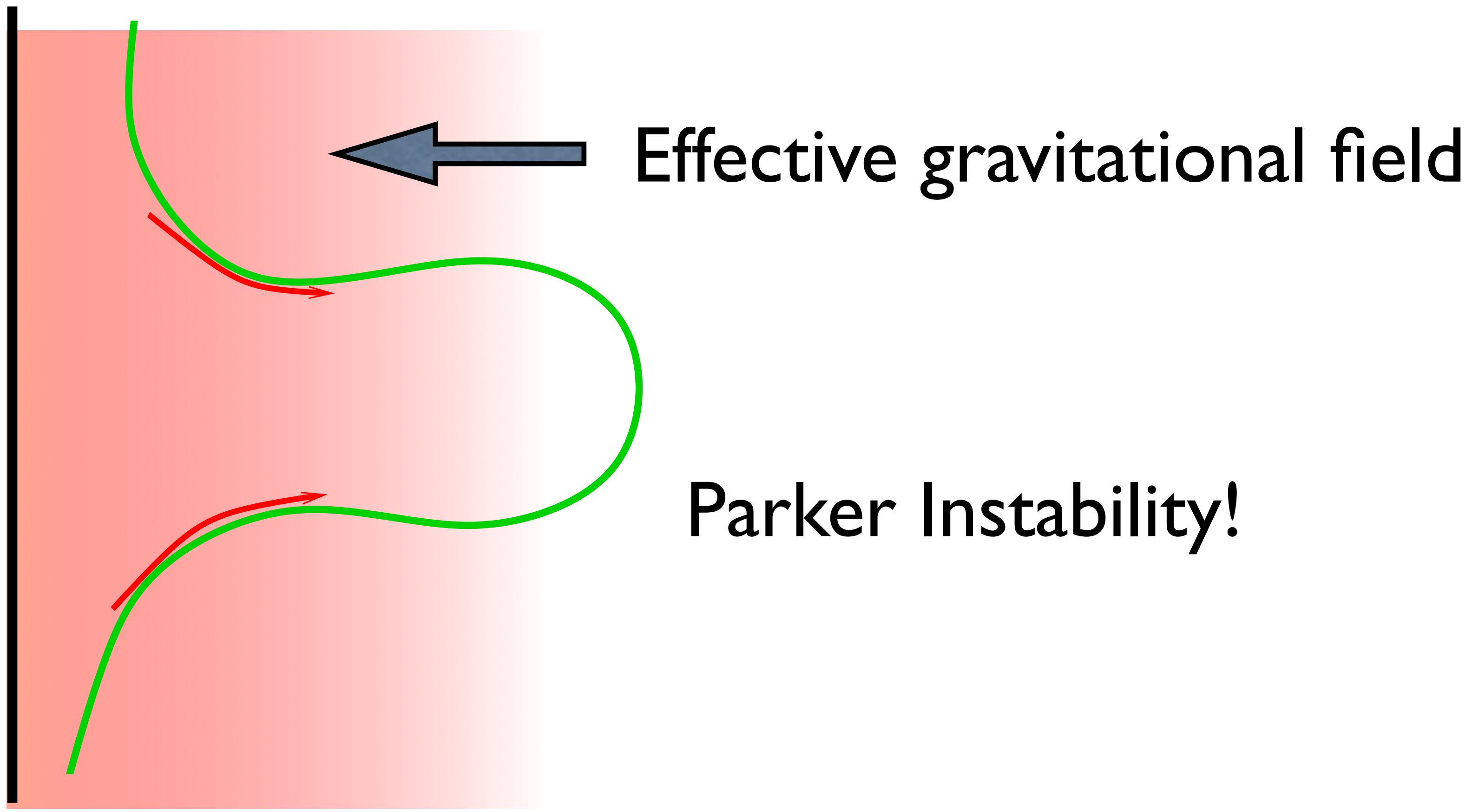
(drive constant flux through structures)

Associated acceleration fluctuations are

$$\frac{d^2 x}{dt^2} \propto \frac{\nabla P_C}{\rho} \propto \frac{1}{\kappa \rho}$$

Unless the diffusion is exactly inversely proportional to the density (two body scattering case!) density fluctuations induce velocity fluctuation which rapidly amplify the initial perturbation. Mathematically appears as a mechanism driving sound waves unstable.

But in some sense not very physical as clearly relies strongly on the 1-D nature of the initial perturbation. Consider more realistic 2-D and 3-D fluctuations in a shock precursor.



- Original Parker instability
  - cosmic ray sources in disc
  - gravitational field of Galaxy
  - buoyant flux tubes inflate with cosmic rays and rise up

- Exactly the same physics should occur in a shock precursor
  - deceleration provides an effective gravitational field towards the shock
  - strong cosmic ray gradient away from shock
  - magnetic field loops linked to the shock should inflate and “rise up”.

Only way to avoid this effect would be to completely decouple diffusion from the density (and magnetic field!) distribution.

$$\frac{\partial \kappa}{\partial \rho} = 0$$

Thus it is impossible for the cosmic-ray pressure gradient to uniformly decelerate density fluctuations in the inflowing plasma and at the same time avoid inducing transverse velocity perturbations.

- Bell's non-resonant instability
  - Energetic cosmic rays penetrate upstream ignoring small-scale field structure - unmagnetised on these scales.
  - Return current of low-energy particles is forced through magnetised plasma
  - Field lines coil up and form helical structures around plasma voids.

- MHD is theory of strongly magnetised plasmas.
- Usually rewrite Lorentz force on plasma to eliminate currents using induction law.
- Need to modify this for case considered.

$$
\begin{aligned}
\nabla \wedge B &= j_{CR} + j_{th} \\
j_{th} &= \nabla \wedge B - j_{CR} \\
F &= j_{th} \wedge B \\
&= (\nabla \wedge B) \wedge B - j_{CR} \wedge B
\end{aligned}
$$

- If CR are strongly scattered then the additional force term can be shown to reduce to an additional cosmic ray pressure,

$$\langle j_{CR} \wedge B \rangle \approx \nabla P_{CR}$$

- But on scales where the CR are not scattered appears as a current driven magnetic field instability.

Figure 4: The paths of four magnetic field lines in (x,y,z) at $t = 6$.

(from Bell 2005)

Good recent account and confirmation of Bell's magnetic field amplification process in arXiv0801.4486 by Zirakashvili, Ptuskin and Voelk.

Instability driven by the cosmic ray diffusion current causes spirals of magnetic field to expand and interact.

Turbulence and multiple local MHD shocks in precursor!

Field saturates at about

$$\rho U^2 \left(\frac{U}{c}\right) \propto \rho V^3$$

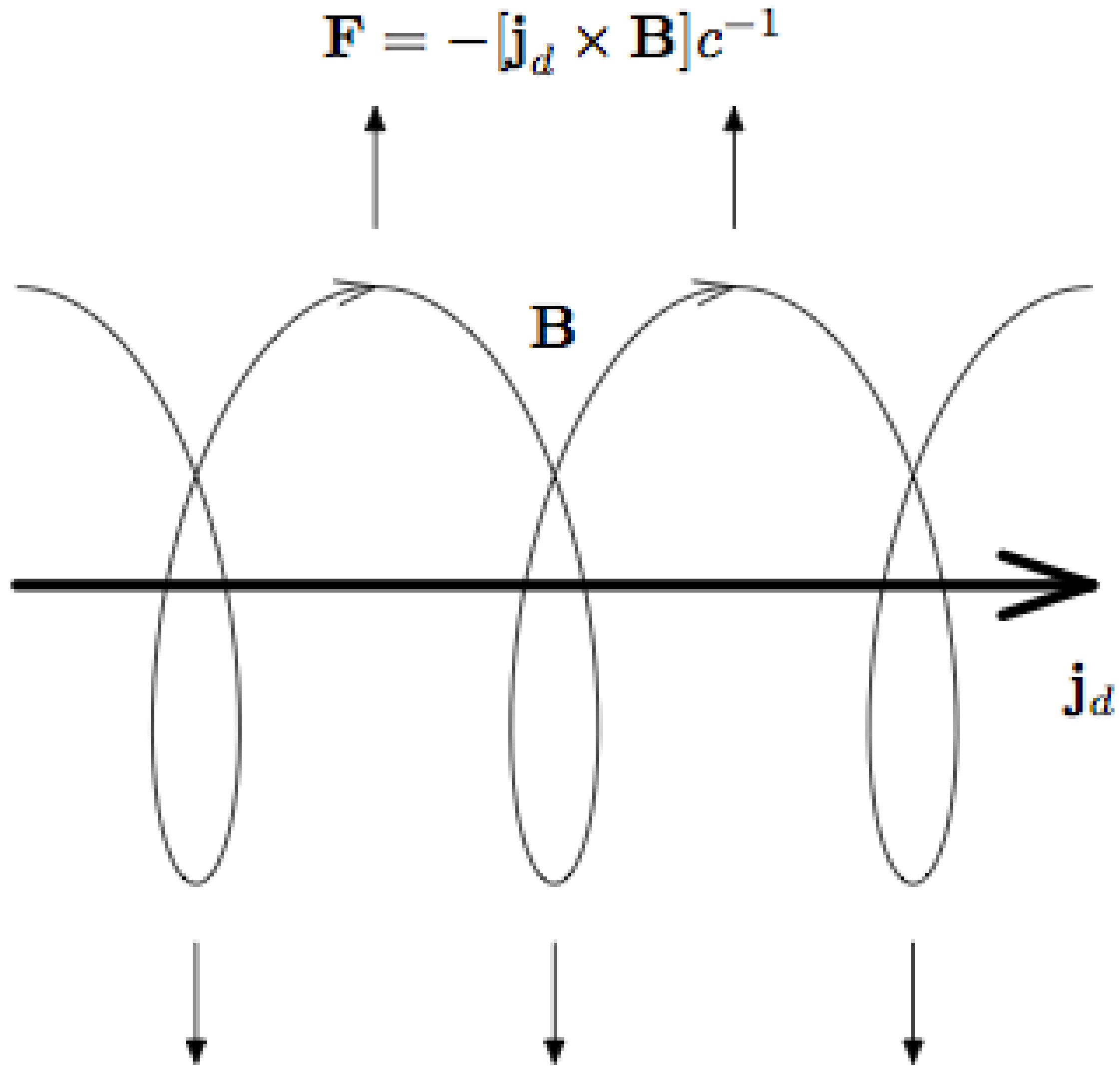


Fig. 1.— Explanation of the non-resonant streaming instability. The magnetic spiral (thin solid line) is stretched by the Lorentz force $\mathbf{F} = -[\mathbf{j}_\mathrm{d} \times \mathbf{B}]c^{-1}$ that appears due to the diffusive cosmic ray electric current $\mathbf{j}_\mathrm{d}$.

Summary

- Shock precursors are almost certainly highly “turbulent” - not clear what implications if any this has for the modification theories.
- Easy to amplify small-scale magnetic field by stretching and twisting.
- Field can plausibly be increased by orders of magnitude, if not to equipartition (Bell predicts saturation a factor U/c below).

- Strong observational indications of amplified fields in young SNRs (with possible U^3 scaling according to J Vink).

- Allows acceleration of protons to “knee region” with ease - otherwise a bit difficult (but scale issue?).

- NB - upstream amplification is needed to reach higher energies, but the observational evidence is only for downstream fields!

Relativistic shocks

See arXiv:0807.3459 by Pelletier, Lemoine and Marcowith for a good recent account.

- In principle the same basic acceleration process, multiple shock crossings with magnetostatic scattering on either side, should work.
- But there are a number of major differences as well as at least one serious problem.

- Distributions are highly anistropic at relativistic shocks, and the energy changes are not small - relativistic shock acceleration is certainly not diffusive!

- Relativistic shocks with magnetic fields are generically superluminal - not clear that particles can in fact recross the shock if they are at all magnetised, or that they have time to be scattered in angle before being overtaken while upstream.

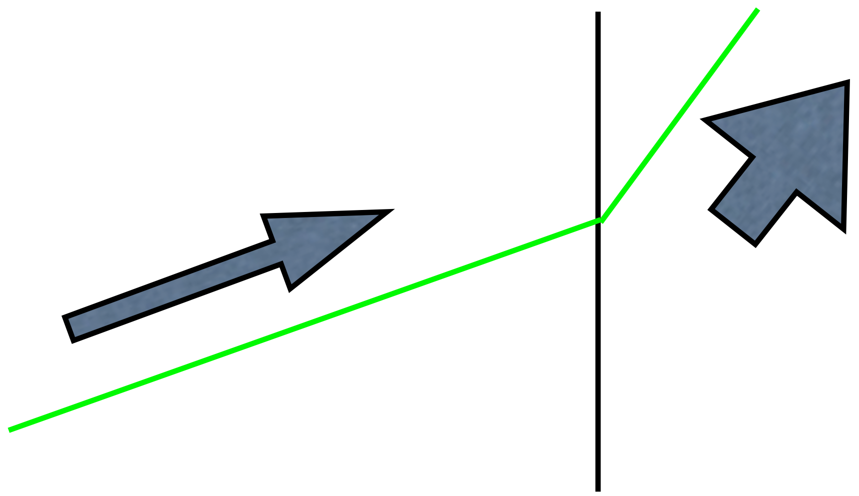
- If large scale regular field with well defined field lines, can divide shocks into those where the point of intersection between shock front and field line moves at less than the speed of light (sub-luminal shocks) and those where it moves faster (super-luminal shocks).

- In sub-luminal case can boost into de Hoffman-Teller frame where this point is at rest.

- In super-luminal case can boost to infinite velocity - ie field is strictly perpendicular

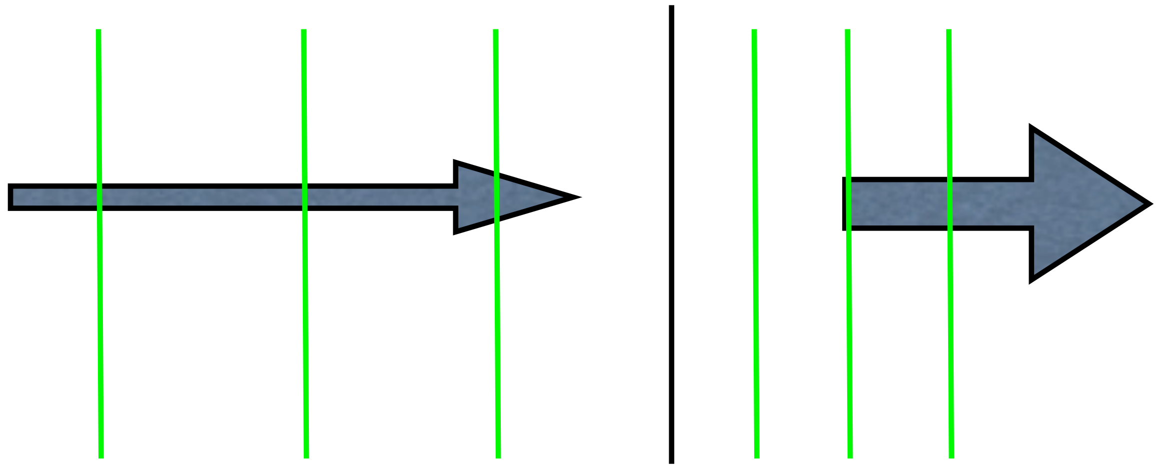
Sub-luminal case

$U \parallel B$



Super-luminal case

$U \perp B$



- If particles are tied to field lines clear that superluminal shocks are not good sites for Fermi acceleration - requires fast cross-field diffusion and strong turbulence (or very random field).

- NB cross-field diffusion at perpendicular shocks can give Fermi acceleration at low speed non-relativistic shocks (Jokipii).

- Hard to see how this could work for strongly magnetised relativistic shocks though.

- However, if one ignores these issues and just assumes that a Fermi type acceleration occurs, then general agreement that a universal spectrum with exponent about 4.2 to 4.3 is formed.

- Recent development is very promising work by Anatoly Spitkovsky on pure electron positron shocks with self-generated Weibel fields where he see first direct evidence for Fermi acceleration in relativistic PIC simulations.

- Suggests (J Arons) that the solution to relativistic shock acceleration is that you need unmagnetised upstream media (or at least ones where the field is weak enough that the Weibel fields can dominate).

- However some form of relativistic shock acceleration is clearly needed for GRB models as well as pulsar wind nebulae.