1) Radiation models

$$\frac{\partial N_i}{\partial t} = \frac{\partial}{\partial E}\left(-b(E)N_i(E)\right) - \frac{N_i(E)}{t_{esc}} + Q_i(E) \quad 0)$$

(1)

$\rightarrow$ Steady state $\partial N/\partial t = 0$

$$Q_i(E) = \frac{\partial}{\partial E}\left(b(E)N_i(E)\right) + \frac{N_i(E)}{t_{esc}(E)} \quad 1)$$

injection = energy losses + escape, $b(E) = -E\,t_{cool}^{-1}$

$$t_{cool}^{-1} = \frac{1}{E}\frac{dE}{dt}$$

$$\left[\rightarrow b(E) = -\frac{dE}{dt}\right]$$

a) Origin of cooling term?

Use total number $\tilde{N}\left[\frac{1}{cm^3}\right]$ [sloppy]

$$\tilde{N}(E, t_0+\Delta t) = \tilde{N}(E,t_0) + \Delta t \cdot \frac{\partial \tilde{N}}{\partial t}(E(t))\Big|_{t_0}$$

[no explicit time dep. $\rightarrow$ cooling term speculate!]

$$= \tilde{N}(E,t_0) + \Delta t \cdot \frac{\partial \tilde{N}}{\partial E}\cdot\frac{\partial E}{\partial t}\Big|_{t_0} \quad \Big|\frac{\partial}{\partial E}$$

$$N(E,t_0+\Delta t) = N(E,t_0) - \Delta t \cdot \frac{\partial}{\partial E}\left(N\,b(E)\right)\Big|_{t_0} \qquad N\left[\frac{1}{cm^3}\frac{1}{GeV}\right]$$

$$\Rightarrow \frac{\partial N}{\partial t} = -\frac{\partial}{\partial E}\left(N\,b(E)\right)$$

[$\rightarrow$ time-dep. equation
$\rightarrow$ steady state]

b) Steady state, general solution

Eq. (1) can be analytically solved

- $$N_i(E) = \frac{1}{b(E)}\int^{E} dE'\,Q(E')\exp\left(-\int_{E'}^{E}\frac{dE''}{t_{esc}(E'')\,b(E'')}\right) \quad 2)$$

  if $b(E) \neq 0$

- $$N_i(E) = Q_i(E)\cdot t_{esc}(E) \quad \text{if } b(E) \equiv 0 \quad 3)$$

Special case: $t_{esc} \equiv \infty$ (particle trapped, but loses energy)

$$N_i(E) = \frac{1}{b(E)}\int^{E} dE'\,Q(E') \quad 4)$$

c) Free-streaming particle, time-dependent case

$b(E) \equiv 0$, $t_{esc}^{-1} = \frac{c}{R}$ R: size of region

3) $\Rightarrow$ $N_i(E) = Q_i(E)\cdot\frac{R}{c}$ 5) in steady state $\left[N: \frac{1}{cm^3 eV}\quad Q: \frac{1}{cm^3 eV s}\right]$

$\uparrow$ injection

Escape spectrum: $Q_{esc,i} = \frac{N_i(E)}{t_{esc}} = Q_i(E)$ same as injection

Time-dependence: use Eq. (0) $\frac{\partial N_i}{\partial t} = -\frac{N_i(E,t)\,c}{R} + Q_i(E)$

[linear D.E. with inhomogeneity $\rightarrow$ generic sol of hom. equ. + part. solution]

Hom. D.E.: separation of variables

$$\frac{\partial N}{N} = -\partial t\,\frac{c}{R} \Rightarrow \ln N = -\frac{c}{R}t + C$$

$$N = \tilde{C}\,e^{-\frac{c}{R}\cdot t}$$

Inhom. D.E.: 5) is a part. solution!

$$\Rightarrow N_i(t) = Q_i(E)\,\frac{R}{c} + \tilde{C}\cdot e^{-\frac{c}{R}\cdot t}$$

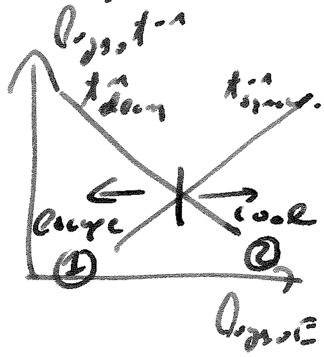
Boundary cond.: $N_i(0) = 0 \Rightarrow \tilde{C} = -\frac{R}{c}\,Q_i(E) \Rightarrow$

$$\Rightarrow \boxed{N_i(t) = Q_i(E)\cdot\frac{R}{c}\left(1 - e^{-\frac{ct}{R}}\right)} \qquad 6)$$

Steady state is reached exponentially quickly in time!

d) Origin of spectral breaks   Example: Pion cooling (for pions)

Solve piece-wise



$Q(E) \propto E^{-\alpha}$; $t_{decay}^{-1} = \frac{m}{\tau_0 \cdot E} \sim \frac{1}{E}$; $t_{synchr}^{-1} \propto \frac{q^4 B^2 E}{m^4} \sim E$

1   3) $\Rightarrow$ $N(E) = Q(E)\cdot t_{decay} \propto \underline{\underline{E^{-\alpha+1}}}$

2   4) $\Rightarrow$ $N(E) = \frac{1}{b(E)}\int^E dE'\,Q(E') \propto \frac{1}{E^2}\cdot E^{-\alpha+1} \propto \underline{\underline{E^{-\alpha-1}}}$

$\nearrow \propto E\cdot E$

Spectral break by two units

NB: The neutrino spectrum follows $Q_{\pi}^{\nu} = \frac{N_\pi(E)}{t_{decay}(E)} \sim \begin{cases} E^{-\alpha} & 1 \\ E^{-\alpha-2} & 2 \end{cases}$

2) Hadronic particle interactions

(3)

Injection from $j \to i$

$$Q_{ji}(E_i) = \int dE_j \, N_j(E_j) \, \Gamma_j(E_j) \, \frac{dn_{j\to i}}{dE_i}(E_j, E_i) \qquad 1) \qquad \left[\to \frac{1}{\text{GeV}\,\text{cm}^3\,\text{s}}\right]$$

primary density $\left[\frac{1}{\text{GeV}}\frac{1}{\text{cm}^3}\right]$; interaction rate $\left[\frac{1}{s}\right]$; re-distribution function $\left[\frac{1}{\text{GeV}}\right]$

Typically $\int \frac{dn_{j\to i}}{dE_i}(E_j, E_i)\, dE_i = M_{j\to i}$ average no of secondaries per interaction, multiplicity 2)

[Ex: $p + \gamma \to \pi^+$ $\frac{1}{3}$ of all cases $\to M_{p\to\pi^+} = \frac{1}{3}$]

a) Feynman scaling   for high energies → low energy target

$$\frac{dn(E_j, E_i)}{dE_i} \rightsquigarrow M_{j\to i}(E_j)\, \rho\Big(\underbrace{\frac{E_i}{E_j}}_{\equiv x}, \sqrt{s}\Big) \qquad 3)$$

prob. distribution normalized to one

Approximation for power law spectra: often:

$$\frac{dn(E_j, E_i)}{dE_i} \simeq \delta(E_i - \chi E_j)\, M_{j\to i} \qquad 4)$$

secondary takes fraction χ of primary energy

$$\underset{1)}{\Rightarrow} Q_{ji}(E_i) = \frac{1}{\chi} N_j\left(\frac{E_i}{\chi}\right) \Gamma_j\left(\frac{E_i}{\chi}\right) M_{j\to i} \qquad 5)$$

from δ-distr.   int. rate at high energy

Consider $N_j(E_j) \propto E_j^{-k} \underset{5)}{\rightsquigarrow} Q_{ji}(E_i) \propto \left(\frac{E_i}{\chi}\right)^{-k}\frac{1}{\chi} \propto E_i^{-k}\chi^{k-1}$

→ same power law, suppressed by factor χ^{k-1} (as $k>1$, $\chi<1$ typically)

b) Weak decays

Rewrite Eq. (1) in terms of $x \equiv \frac{E_i}{E_j} \Rightarrow dx = -\frac{E_i}{E_j^2}\, dE_j = -x\, dE_j/E_j$

$$\Rightarrow \frac{dx}{x} = -\frac{dE_j}{E_j}\ ; \quad dE_i = E_j\, dx \qquad 6)$$

$$\underset{1)}{\Rightarrow} Q_{ji}(E_i) = \int \frac{dE_j}{E_j} N_j(E_j)\, \Gamma_j(E_j)\, E_j \, \frac{dn_{j\to i}(E_j, E_i)}{dE_i}$$

$$= \int \frac{dx}{x}\, N_j\!\left(\frac{E_i}{x}\right) \Gamma_j\!\left(\frac{E_i}{x}\right) \boxed{\overbrace{\frac{dn_{j\to i}(E_i, x)}{dx}}^{= F_{j\to i}\left(x, \frac{E_i}{x}\right)}} \qquad 7)$$

Weak decays: $\Gamma_j\left(\frac{E_i}{x}\right) = t^{-1}_{\text{decay}}\left(\frac{E_i}{x}\right) = \frac{m_j\, x}{\tau_0 E_i}$ → easy to get!

For example: $f_{\pi\to\nu_\mu}(x) = \frac{1}{1-r_\pi}\, \underbrace{\Theta(1-r_\pi - x)}_{\text{step fct. } (>0 \text{ for } x<1-r_\pi)}$ $\quad r_\pi = \left(\frac{m_\mu}{m_\pi}\right)^2$

see arXiv:0704.0718 (Lipari, Lusignoli, Meloni, 2007) for others!

c) pp Interactions

$\Gamma \propto c\, n_H \cdot \sigma_{pp}$ Target density ↓ 1 Primary density ↓

7) ⇒ e.g. $Q_\nu(E_\nu) = c\, n_H \int_0^1 \sigma_{pp}\left(\frac{E_\nu}{x}\right) N_p\left(\frac{E_\nu}{x}\right) \cdot \underbrace{F_{j\to i}\left(x, \frac{E_\nu}{x}\right)}$ 8)

Parametrization → see Eqs. (62) + (66) in Kelner, Aharonian, Bugayov 2006 (astro-ph/0606058)

[pγ interactions more complicated → exercise!]

d) Relationship among ν, γ, n spectra

$$p+\gamma \to \Delta^+ \to \begin{cases} n + \pi^+ & 1/3 \\ p + \pi^0 & 2/3 \end{cases}$$

– Model: $\pi^+ \to \mu^+ + \nu_\mu$, $\hookrightarrow e^+ + \nu_e + \bar{\nu}_\mu$; $\pi^0 \to \gamma + \gamma$

Pion takes ~20% of p energy → Neutrino $x_\nu \sim 0.2 \cdot \frac{1}{4} \sim 0.05$ per flavor
Gamma $x_\gamma \sim 0.2 \cdot \frac{1}{2} \sim 0.1$ per γ

Use Eq. (5)

$$\to Q_{p\to\nu} = N_p\left(\frac{E_\nu}{0.05}\right) \cdot \frac{1}{0.05} \cdot \Gamma\left(\frac{E_\nu}{0.05}\right) \cdot \frac{1}{3} \qquad 9)$$

[sometimes others used $\frac{1}{2}\cdot\frac{1}{2}$!]

$$Q_{p\to\gamma} = N_p\left(\frac{E_\gamma}{0.1}\right) \cdot \frac{1}{0.1}\, \Gamma\left(\frac{E_\gamma}{0.1}\right) \cdot \frac{2}{3} \qquad 10)$$

$$Q_{p\to n} = N_p\left(\frac{E_n}{0.8}\right) \cdot \frac{1}{0.8} \cdot \Gamma\left(\frac{E_n}{0.8}\right) \cdot \frac{1}{3} \qquad 11)$$

⇒ (9),(10) $Q_{p\to\nu}(E_\nu = 0.5 E_\gamma) = Q_{p\to\gamma}(E_\gamma) \;\Big|\cdot E_\gamma^2 \;\Rightarrow\; E_\gamma^2 Q_{p\to\gamma}(E_\gamma) = 4 E_\nu^2 Q_{p\to\nu}\Big|_{E_\nu = 0.5 E_\gamma}$ 12)

[sometimes factor 2 if BR ~ $\frac{1}{2}$ – $\frac{1}{2}$]

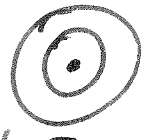
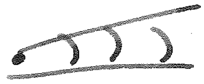
⇒ (10,11) Exercise: derive similar relationships for ~~p→~~γ, n→ν!

3) Energetics of sources + geometry estimates → Example GRBs

We need N_p', N_γ' in source (prime: shock rest frame SRF) 

$$\text{Density} \approx \frac{\text{Energy (total)}}{\text{Energy (particle, typical)} \cdot \text{Volume}} \sim \frac{\text{no of particles}}{\text{volume}} \quad 1)$$

→ Need volume estimators

→ Assume isotropic emission  vs. 

4π — jet solid angle

→ Isotropic energy, isotropic volume

→ Cancels in density!

a) Isotropic volume

$\Delta d'$ shell width

shells collide

Obs

R_c collision radius

$$V'_{iso} = 4\pi R_c^2 \Delta d' \quad 2)$$

$$R_c \simeq 2\Gamma^2 c \frac{t_v}{(1+z)} \quad 3) \text{ from geometry}$$

estimate w. coll. models [angular spreading time = t_v width]

$$\Delta d' \simeq \Gamma \cdot c \cdot \frac{t_v}{(1+z)} \quad 4) \text{ from } t_v \text{ boosted into SRF} \text{ [causality argument]}$$

$$V'_{iso} = 4\pi R_c^2 \Delta d' = 16\pi \Gamma^5 c^3 \frac{t_v^3}{(1+z)^3} \quad 5) \text{ strongly depends on } \Gamma!$$

[actually Doppler factor instead of Γ, which dep. on obs. angle!]

b) From observed energy to energy in SRF

Energy of GRB is typically specified in engine frame (e.g. black hole)

E_{iso}: isotropic equivalent energy (in γ-rays) [erg]

L_{iso}: isotropic luminosity [erg/s]

where $L_{iso} = \dfrac{E_{iso}}{T_{90}/(1+z)}$ 6) [T_{90}: time in which 90% of photons arrive, has to be blue-shifted from obs. frame]

Observable: bolometric equivalent energy S_{bol} [erg/cm²] [corrected for energy band]

$$\rightarrow E_{iso} = \frac{4\pi d_L^2}{(1+z)} S_{bol} \quad 7)$$

[$4\pi d_L^2$ scales luminosity w. lum. distance] [Interpret that this emission!]

$[\Rightarrow L_{iso} = 4\pi d_L^2 \left(\frac{S_{bol}}{T_{90}}\right)_{\text{avg. flux}}$ consistent with Eq. (6)]

→ Energy in SRF per collision

$$E'_{iso,sh} = \frac{E_{iso}}{\Gamma} \cdot \frac{1}{N} \quad 8)$$

N: no. of collisions $\approx \frac{T_{90}}{t_v}$

c) Normalization of densities

$$\underset{\text{energy density}}{U'_\gamma} = \underbrace{\int \varepsilon' N'_\gamma(\varepsilon') \, d\varepsilon'}_{\left[\frac{\text{erg}}{\text{cm}^3}\text{ ph. energy density}\right]} \doteq \frac{E'_{iso,sh}}{V'_{iso}} \quad 9)$$

[Γ et quad cancels]

⇒ Normalization of $N'_\gamma(\varepsilon')$
[Shape from obs.: broken power law]

Protons: $\int E'_p N'_p(E'_p) \, dE'_p \doteq \frac{E'_{iso,sh}}{V'_{iso}} \cdot \boxed{\frac{1}{f_e}}$ 10)

baryonic loading (assumption)
~10-100 from UHECR normal.

Magnetic field: $U'_B \doteq \frac{1}{f_B} \frac{E'_{iso,sh}}{V'_{iso}}$ 11)

magn. loading ~1

→ Can compute particle interactions

d) Pion production efficiency (estimate)

$$f_\pi \approx \frac{\Delta d'}{\lambda'_{p\gamma}} \langle x_{p\to\pi} \rangle \quad 12)$$

fraction of proton energy dumped into pion production

mean free path; average fraction per interaction ~0.2

$\lambda'_{p\gamma} \sim \frac{1}{N'_\gamma \sigma_\Delta}$, $N'_\gamma \approx \frac{U'_\gamma}{\varepsilon'_{\gamma,br}}$ (→ Eq. 9)

typical photon energy

$$\Rightarrow f_\pi = [\ldots] \propto L_{iso} \frac{1}{\Gamma^4 t_v} \frac{1}{\varepsilon_{\gamma,br}}$$

(obs. frame)

Strong dependence on observables, as strong dependence on geometry estimates

[Small Γ, small volume for same energy in SRF
→ high densities
→ a lot of pion production]

Ref.: Guetta et al. 2004