The Cosmological Standard Model
and its Parameters

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Geometry and Dynamics
Structure

1. Geometry and Dynamics
2. Parameters, Age and Distances
3. Thermal Evolution
4. Recombination and Nucleosynthesis
5. The Growth of Perturbations
6. Statistics and Non-linear Evolution
7. Structures in the Cosmic Microwave Background
8. Cosmological Inflation
9. Dark Energy
Geometry and Dynamics

1. Assumptions
2. Metric
3. Redshift
4. Dynamics
5. Remarks on Newtonian Dynamics
cosmology rests on two fundamental assumptions:

1. when averaged over sufficiently large scales, the observable properties of the Universe are isotropic
2. our position in the Universe is by no means preferred to any other (cosmological principle);

such a Universe is *homogeneous and isotropic*

only relevant interaction is gravity, thus we search for cosmological models in General Relativity
Galaxy Distribution

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The Cosmological Standard Model and its Parameters
due to symmetry, the $4 \times 4$ tensor $g_{\mu \nu}$ has ten independent components: $g_{00}$, $g_{0i}$, and $g_{ij}$; the two fundamental assumptions greatly simplify the metric
eigentime should equal coordinate time for fundamental observers:

$$ds^2 = g_{00}dt^2 = c^2dt^2 \Rightarrow g_{00} = c^2 \quad (1)$$

isotropy requires $g_{0i} = 0$ and spherical symmetry for three-space, thus

$$ds^2 = c^2dt^2 - a^2(t)\left[ dw^2 + f_K^2(w)d\omega^2 \right], \text{ with} \quad (2)$$

$$f_K(w) = \begin{cases} 
K^{-1/2} \sin(K^{1/2}w) & (K > 0) \\
w & (K = 0) \\
|K|^{-1/2} \sinh(|K|^{1/2}w) & (K < 0) 
\end{cases} \quad (3)$$
Symmetry, Foliation
spatial hypersurfaces can expand or shrink, leading to red- or blueshift

the propagation condition for light, \( ds = 0 \), leads to

\[
\frac{\nu_e}{\nu_o} = \frac{\lambda_o}{\lambda_e} = 1 + \frac{\lambda_o - \lambda_e}{\lambda_e} = 1 + z = \frac{a(t_e)}{a(t_o)}
\]

thus, light is red- or blueshifted by the same amount as the Universe expanded or shrunk between emission and observation
the dynamics of the metric (2) is reduced to the dynamics of the scale factor $a(t)$; differential equations for $a(t)$ now follow from Einstein’s field equations:

$$
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{Kc^2}{a^2} + \frac{\Lambda}{3}, \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \left( \rho + \frac{3p}{c^2} \right) + \frac{\Lambda}{3}
$$

Friedmann’s equations

the Friedmann equations can be combined to yield the adiabatic equation

$$
\frac{d}{dt} \left( a^3 \rho c^2 \right) + p \frac{d}{dt} \left( a^3 \right) = 0
$$

energy conservation
Remark on Newtonian Dynamics

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homogenes, isotropes Universum

kugelförmiger Ausschnitt

Dichte $\rho$, Radius R
Remark on Newtonian Dynamics

- (5) can also be derived from Newtonian gravity, except for the $\Lambda$ term;
- study homogeneous sphere of radius $r$, ignore surrounding matter; size of the sphere is arbitrary
- pressure term adds to the density because pressure means kinetic energy of particles, which is equivalent to a mass density; yields equation of motion

$$\ddot{r} = -\frac{4\pi G}{3} r \left( \rho + \frac{3p}{c^2} \right)$$

(7)

- integrating, using energy conservation, we find

$$\left( \frac{\dot{r}}{r} \right)^2 = \frac{8\pi G}{3} \rho + \frac{C}{c^2}$$

(8)
Parameters, Age and Distances
Parameters, Age and Distances

1. Forms of Matter
2. Parameters
3. Parameter Values
4. Age and Expansion of the Universe
5. Distances
6. Horizons
two forms of matter can broadly be distinguished, relativistic and non-relativistic; they are often called radiation and dust, respectively

for relativistic bosons and fermions:

\[ p = \frac{\rho c^2}{3} \]  \hspace{1cm} (9)

for which (6) implies

\[ \rho(t) = \rho_0 a^{-4} , \]  \hspace{1cm} (10)

\( a = 1 \) today

for non-relativistic matter, \( p = 0 \) because \( p \ll \rho c^2 \), and

\[ \rho(t) = \rho_0 a^{-3} \]  \hspace{1cm} (11)
### Parameters (1)

- **Hubble parameter**, relative expansion rate:

  \[
  H(t) \equiv \frac{\dot{a}}{a} , \quad H_0 \equiv H(t_0) = 100 \, h \, \frac{\text{km}}{\text{s Mpc}} = 3.2 \times 10^{-18} \, h \, \text{s}^{-1}
  \]  
  \[\text{(12)}\]

- **Critical density**

  \[
  \rho_{\text{cr}}(t) \equiv \frac{3H^2(t)}{8\pi G} , \quad \rho_{\text{cr}0} \equiv \rho_{\text{cr}}(t_0) = \frac{3H_0^2}{8\pi G} = 1.9 \times 10^{-29} \, h^2 \, \text{g cm}^{-3}
  \]  
  \[\text{(13)}\]

- **Dimensionless density parameters**

  \[
  \Omega(t) \equiv \frac{\rho(t)}{\rho_{\text{cr}}(t)} , \quad \Omega_0 \equiv \frac{\rho(t_0)}{\rho_{\text{cr}0}} , \quad \Omega_{\Lambda}(t) = \frac{\Lambda}{3H^2(t)} , \quad \Omega_{\Lambda0} \equiv \frac{\Lambda}{3H_0^2}
  \]  
  \[\text{(14)}\]
Friedmann’s equation becomes

\[ H^2(a) = H_0^2 \left[ \Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_\Lambda - \frac{Kc^2}{a^2} \right] \]  \hspace{1cm} (15)

specialising to \( a = 1 \) allows to solve for \( K \),

\[ -Kc^2 = 1 - \Omega_r - \Omega_m - \Omega_\Lambda \equiv \Omega_K \]  \hspace{1cm} (16)

final form for Friedmann’s equation

\[ H^2(a) = H_0^2 \left[ \Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_\Lambda + \Omega_K a^{-2} \right] \equiv H_0^2 E^2(a) \]  \hspace{1cm} (17)
radiation density exceeded matter density before

\[ a_{eq} = \frac{\Omega_{r0}}{\Omega_{m0}} \]  \hspace{1cm} (18)

the density parameters change with time:

\[ \Omega_m(a) = \frac{\Omega_{m0}}{a + \Omega_{m0}(1 - a) + \Omega_{\Lambda0}(a^3 - a)} , \]

\[ \Omega_\Lambda(a) = \frac{\Omega_{\Lambda0}a^3}{a + \Omega_{m0}(1 - a) + \Omega_{\Lambda0}(a^3 - a)} \]  \hspace{1cm} (19)

this implies: \( \Omega_m(a) \rightarrow 1 \) and \( \Omega_\Lambda(a) \rightarrow 0 \) for \( a \rightarrow 0 \) regardless of their present values; if \( \Omega_{m0} + \Omega_{\Lambda0} = 1 \), remains valid for \( a < 1 \)
## Parameter Values

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hubble constant</td>
<td>$h$</td>
<td>$0.70^{+0.04}_{-0.03}$</td>
<td>CMB + SDSS HST Key Project</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$0.72 \pm 0.07$</td>
<td></td>
</tr>
<tr>
<td>Matter density</td>
<td>$\Omega_{m0}$</td>
<td>$0.30 \pm 0.04$</td>
<td>assuming $\Omega_K = 0$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>free $\Omega_K$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$0.41 \pm 0.09$</td>
<td></td>
</tr>
<tr>
<td>Cosmological constant</td>
<td>$\Omega_{\Lambda 0}$</td>
<td>$0.70 \pm 0.04$</td>
<td>assuming $\Omega_K = 0$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>free $\Omega_K$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$0.65 \pm 0.08$</td>
<td></td>
</tr>
<tr>
<td>Curvature</td>
<td>$\Omega_K$</td>
<td>$-0.06 \pm 0.04$</td>
<td>free $\Omega_K$</td>
</tr>
<tr>
<td>Baryon density</td>
<td>$h^2\Omega_B$</td>
<td>$0.023 \pm 0.001$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\Omega_B$</td>
<td>$0.047 \pm 0.006$</td>
<td></td>
</tr>
<tr>
<td>Radiation density</td>
<td>$\Omega_{r0}$</td>
<td>$(2.494 \pm 0.007) \cdot 10^{-5}$</td>
<td>from CMB temperature</td>
</tr>
<tr>
<td>Hubble time</td>
<td>$H_0^{-1}$</td>
<td>$(1.4 \pm 0.08) \times 10^{10}$ yr</td>
<td></td>
</tr>
<tr>
<td>Matter-radiation equality</td>
<td>$a_{eq}$</td>
<td>$(8.3 \pm 1.1) \times 10^{-5}$</td>
<td></td>
</tr>
</tbody>
</table>
since $H = \dot{a}/a$, the age of the Universe is determined by

$$\frac{da}{dt} = H_0 a E(a) \Rightarrow H_0 t = \int_0^a \frac{da'}{a'E(a')} \quad (20)$$

in a flat universe with $\Omega_{m0} \neq 0$ and $\Omega_\Lambda = 1 - \Omega_{m0} \neq 0$:

$$H_0 t = \frac{2}{3 \sqrt{1 - \Omega_{m0}}} \arcsinh \left[ \sqrt{\frac{1 - \Omega_{m0}}{\Omega_{m0}}} a^{3/2} \right] \quad (21)$$

the age of our universe is

$$t(a = 1) = \frac{0.96}{H_0} = 1.35 \times 10^{10} \text{ yr} \quad (22)$$
the Universe should be older than its oldest parts

three ways of measuring the ages:

1. nuclear cosmo-chronology: decay of long-lived nuclei;
   \[ \approx 4.6 \text{ Gyr for the Earth, } 7 \ldots 13 \text{ Gyr for the Galaxy}; \]

2. ages from stellar evolution: \( \gtrsim 13 \text{ Gyr from globular clusters}; \)

3. cooling of white dwarfs: \( \approx 10 \text{ Gyr} \)

\[ t(a = 1) \gtrsim 11 \text{ Gyr needs } H_0 \lesssim 61 \text{ km s}^{-1} \text{ Mpc}^{-1} \text{ in an Einstein-de Sitter universe} \]
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distance measures are no longer unique in general relativity

**proper distance** $D_{\text{prop}}$, $dD_{\text{prop}} = -cdt = -cd\dot{a}/\dot{a}$

**comoving distance** $D_{\text{com}}$, $dD_{\text{com}} = dw$

**angular diameter distance** $D_{\text{ang}}$

\[
D_{\text{ang}}(z_1, z_2) = \left(\frac{\delta A}{\delta \omega}\right)^{1/2} = a(z_2)f_K[w(z_1, z_2)] \tag{23}
\]

**luminosity distance** $D_{\text{lum}}$,

\[
D_{\text{lum}}(z_1, z_2) = \left[\frac{a(z_1)}{a(z_2)}\right]^2 D_{\text{ang}}(z_1, z_2) \tag{24}
\]
Distance Measures

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galaxies move away from us with velocities proportional to their distance; Hubble law,

\[ D = \frac{cz}{H_0} \quad \Rightarrow \quad v = cz = H_0 D ; \] (25)

local deviations due to peculiar velocities
accurate distance measurements to distant objects required
“standard candles”: Cepheids, Supernovae, galaxy scaling relations
result from \textit{Hubble Key Project}:

\[ H_0 = 72 \pm 8 \text{ km s}^{-1} \text{ Mpc}^{-1} \] (26)
Measurements of the Hubble Constant

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The Cosmological Standard Model and its Parameters
Horizons

- between $t_1$ and $t_2 > t_1$, light can travel across comoving distance

$$\Delta w(t_1, t_2) = \int_{t_1}^{t_2} \frac{cdt}{a(t)} = c \int_{a(t_1)}^{a(t_2)} \frac{da}{a\dot{a}} \propto a^{n/2-1} \quad \text{if} \quad \rho \propto \rho_0 a^{-n}$$

(27)

thus, if $n > 2$, light can only travel by a finite distance; there exists a particle horizon

- Hubble radius at $a_{eq}$, important for structure formation

$$r_{H,eq} = \frac{c}{H(a_{eq})} = \frac{c}{H_0} \frac{a_{eq}^{3/2}}{\sqrt{2\Omega_m}}$$

(28)
Chapter III

Thermal Evolution
1. Assumptions
2. Properties of Ideal Quantum Gases
3. Adiabatic Expansion of Ideal Gases
4. Particle Freeze-Out
Assumptions

- *the universe expands adiabatically* – isotropy requires the universe to expand adiathermally; entropy generation is completely negligible
- *thermal equilibrium can be maintained despite the expansion*
- *the cosmic “fluids” can be treated as ideal gases*
- those assumptions are the starting point of our considerations; they need to be verified as we go along
Properties of Ideal Quantum Gases

for relativistic boson and fermion gases in thermal equilibrium:

\[
\begin{align*}
  n_B &= 10 g_B \left( \frac{T}{K} \right)^3 \text{ cm}^{-3} = 1.6 \times 10^{13} g_B \left( \frac{kT}{eV} \right)^3 \text{ cm}^{-3}, \\
  n_F &= \frac{3}{4} \frac{g_F}{g_B} n_B \\
  u_B &= 3.8 \times 10^{-15} g_B \left( \frac{T}{K} \right)^4 \frac{\text{erg}}{\text{cm}^3} = 2.35 \times 10^{-3} g_B \left( \frac{kT}{eV} \right)^4 \frac{\text{erg}}{\text{cm}^3}, \\
  u_F &= \frac{7}{8} \frac{g_F}{g_B} u_B, \quad P_B = \frac{u_B}{3}, \quad P_F = \frac{u_F}{3} \\
  \frac{s_B}{k} &= 36 g_B \left( \frac{T}{K} \right)^3 \text{ cm}^{-3} = 5.7 \times 10^{13} g_B \left( \frac{kT}{eV} \right)^3 \text{ cm}^{-3}, \\
  s_F &= \frac{7}{8} \frac{g_F}{g_B} s_B
\end{align*}
\]
Adiabatic Expansion of Ideal Gases

- for relativistic boson or fermion gases in thermal equilibrium

\[ P = \frac{u}{3} = \frac{E}{3V} \quad (30) \]

- first law of thermodynamics implies

\[ dE = -PdV = 3d(PV) \Rightarrow P \propto V^{-4/3} \quad (31) \]

i.e. \( \gamma = 4/3 \); for non-relativistic ideal gases, \( \gamma = 5/3 \)

- temperature scaling:

\[ T \propto P^{1/4} \propto V^{-1/3} \propto a^{-1} \quad (rel.) \]
\[ T \propto PV \propto V^{-5/3+1} \propto a^{-2} \quad (non-rel.) \quad (32) \]
expansion time-scale during radiation-dominated era

\[ t_{\text{exp}} \approx (G\rho)^{-1/2} \propto a^{-2} \quad (33) \]

collision rate and time-scale

\[ \Gamma \equiv n\langle \sigma v \rangle \propto n \propto T^3 \propto a^{-3} , \quad t_{\text{coll}} = \Gamma^{-1} \propto a^3 \quad (34) \]

ratio \( t_{\text{exp}}/t_{\text{coll}} \propto a^{-1} \), thermal equilibrium can be maintained despite the expansion at early times; as the universe keeps expanding, thermal equilibrium breaks down when \( \Gamma \ll H \)

relativistic particle species retain their thermal-equilibrium density!
Chapter IV

Recombination and Nucleosynthesis
Recombination and Nucleosynthesis

1. Neutrino Background
2. Photons and Baryons
3. Recombination Process
4. Nucleosynthesis
The Neutrino Background

- weak interaction

\[ \nu + \bar{\nu} \leftrightarrow e^+ + e^- \quad (35) \]

freezes out when temperature drops to

\[ T_{\nu} \approx 10^{10.5} \text{ K} \approx 2.7 \text{ MeV} \]

- electron-positron pairs annihilate when temperature drops below \( T \approx 2m_e c^2 \approx 1 \text{ MeV} \approx 10^{10} \text{ K} \) their decay heats the photon gas, but not the neutrinos

- photon temperature is \( \approx 40\% \) higher than neutrino temperature:

\[ T_{\gamma} = \left( \frac{11}{4} \right)^{1/3} T_{\nu} \quad (36) \]

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Photons and Baryons

- The number density of baryons today is

\[ n_B = \frac{\rho_B}{m_p} = \frac{\Omega_B}{m_p} \frac{3H_0^2}{8\pi G} = 1.1 \times 10^{-5} \Omega_B h^2 \text{ cm}^3 \]

\[ \Omega_B h^2 \approx 0.02 \quad (37) \]

- The photon number density today is

\[ n_\gamma = 407 \text{ cm}^{-3} \quad (38) \]

- Their ratio is constant; about one billion photons per baryon!

\[ \eta \equiv \frac{n_B}{n_\gamma} = 2.7 \times 10^{-8} \Omega_B h^2 \quad (39) \]
approximation: Saha's equation; ionisation fraction $x$ is

$$\frac{x^2}{1 - x} = \frac{\sqrt{\pi}}{4 \sqrt{2}\zeta(3)\eta} \left(\frac{m_e c^2}{kT}\right)^{3/2} \exp\left(-\frac{\chi}{kT}\right) \approx \frac{0.26}{\eta} \left(\frac{m_e c^2}{kT}\right)^{3/2} \exp\left(-\frac{\chi}{kT}\right)$$

(40)

for recombination to be considered finished, $x \ll 1$ and $x^2/(1 - x) \approx x^2$; since $1/\eta$ is a huge number, $kT \ll \chi$ is required for $x$ to be small; for example, putting $x = 0.1$ yields $kT_{\text{rec}} = 0.3$ eV, or

$$T_{\text{rec}} \approx 3500 \text{ K}$$

(41)

since $\chi = 13.6$ eV, one would naively expect $T_{\text{rec}} \approx 10^5$ K; the very large photon-to-baryon ratio $1/\eta$ delays recombination considerably
Recombination

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The Spectrum of the CMB

FIRAS data with 400σ errorbars

2.725 K Blackbody

Nobelprize.org

The Nobel Prize in Physics 2006

"for their discovery of the blackbody form and anisotropy of the cosmic microwave background radiation"

John C. Mather
USA
NASA Goddard Space Flight Center
Greenbelt, MD, USA
In 1966

George F. Smoot
USA
University of California
Berkeley, CA, USA
In 1965

Photos: Copyright © The Nobel Foundation
protons and neutrons form when $kT \approx 1 \text{ GeV}$; remain in equilibrium until weak interactions freeze out at $kT \approx 800 \text{ keV}$; baryon-to-photon ratio $\eta$ is the only relevant parameter,

$$\eta = 10^{10} \eta_{10}, \quad \eta_{10} = 273 \Omega_B h^2$$  \hfill (42)

further fusion builds upon two-body processes, e.g.

$$n + p \rightarrow ^2\text{H} + \gamma, \quad ^2\text{H} + ^2\text{H} \rightarrow ^3\text{He} + n, \quad ^3\text{He} + ^2\text{H} \rightarrow ^4\text{He} + p, \quad ^4\text{H} + ^3\text{H} \rightarrow ^7\text{Li} + \gamma$$  \hfill (43)

deuterium is crucial! (Gamow criterion)

the baryon density implied by Big-Bang nucleosynthesis is

$$\Omega_B h^2 = 0.019 \pm 0.0024$$  \hfill (44)
Light-Element Abundances

QSO 1937–1009
$z_{\text{abs}} = 3.572$

Fraction of critical density

0.01  0.02  0.05

$^4\text{He}$ Mass fraction

$^{3}\text{He}$

$^7\text{Li}$

Number relative to H

10^{-6}  10^{-10}

Baryon density ($10^{-31}$ g cm^{-3})

5  2  1

D

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Chapter V

The Growth of Perturbations
The Growth of Perturbations

1. Newtonian Equations
2. Perturbation Equations
3. Density Perturbations
4. Velocity Perturbations
Newtonian Equations

- **continuity equation (mass conservation)**
  \[
  \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \tag{45}
  \]

- **Euler’s equation (momentum conservation)**
  \[
  \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{\nabla p}{\rho} + \nabla \Phi \tag{46}
  \]

- **Poisson equation**
  \[
  \nabla^2 \Phi = 4\pi G \rho \tag{47}
  \]
Perturbation Equations

decompose density and velocity,

\[ \rho(t, \vec{x}) = \rho_0(t) + \delta \rho(t, \vec{x}), \quad \vec{v}(t, \vec{x}) = \vec{v}_0(t) + \delta \vec{v}(t, \vec{x}) \quad (48) \]

Hubble flow, peculiar velocity:

\[ \vec{v} =  \dot{\vec{r}} = \dot{a} \vec{x} + a \dot{\vec{x}} = H \vec{r} + a \dot{\vec{x}} = \vec{v}_0 + \delta \vec{v} \quad (49) \]

comoving coordinates \( \vec{x} = \vec{r}/a \), comoving peculiar velocities \( \vec{u} \equiv \delta \vec{v}/a \), density contrast \( \delta = \delta \rho/\rho_0 \)

we are now left with the three equations

\[ \dot{\delta} + \vec{\nabla} \cdot \vec{u} = 0 \, , \quad \dot{\vec{u}} + H \vec{u} = -\frac{\vec{\nabla} \delta \rho}{a^2 \rho_0} + \frac{\vec{\nabla} \delta \Phi}{a^2} \, , \quad \nabla^2 \delta \Phi = 4\pi G \rho_0 a^2 \delta \quad (50) \]
combining these equations, decomposing $\delta$ into plane waves yields

$$\ddot{\delta} + 2H\dot{\delta} = \delta \left( 4\pi G\rho_0 - \frac{c_s^2 k^2}{a^2} \right)$$

(51)

sound speed $c_s^2 = \delta p/\delta \rho$

assume large perturbations,

$$\frac{c_s^2 k^2}{a^2} \ll 4\pi G\rho_0 \quad \Rightarrow \quad k \ll a \frac{\sqrt{4\pi G\rho_0}}{c_s}$$

(52)

linear growth factor

$$\delta(a) = \delta_0 D_+(a)$$

(53)
Growth Factor

\[ D(z)/a \]

redshift \( z \)

\( \Omega=1, \Lambda=0 \)
\( \Omega=0.3, \Lambda=0 \)
\( \Omega=0.3, \Lambda=0.7 \)

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Structure Growth and Dark Matter

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The Amount of Dark Matter

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ignoring pressure gradients, the second equation (50) says

\[ \dot{\mathbf{u}} + H\mathbf{u} = \mathbf{\nabla}\delta \Phi / a^2 \]  

(54)

defining

\[ f(\Omega) \equiv \frac{d\ln D_+(a)}{d\ln a} \approx \Omega^{0.6} \]  

(55)

the peculiar velocity field can be written as

\[ \delta \mathbf{v} = a\mathbf{\ddot{u}} = \frac{2f(\Omega)}{3aH\Omega} \mathbf{\nabla}\delta \Phi \]  

(56)
The Local Cosmic Neighbourhood

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The Cosmological Standard Model and its Parameters
Chapter VI

Statistics and Non-Linear Evolution
Statistics and Non-Linear Evolution

1. Power Spectra
2. Evolution of the Power Spectrum
3. The Zel’dovich Approximation
4. Nonlinear Evolution
variance of $\delta$ in Fourier space defines the power spectrum $P(k)$,

$$\langle \hat{\delta}(\vec{k})\hat{\delta}^*(\vec{k}') \rangle \equiv (2\pi)^3 P(k) \delta_D(\vec{k} - \vec{k}') \quad (57)$$

variance of $\delta$ in real space depends on scale:

$$\bar{\delta}(\vec{x}) \equiv \int d^3y \delta(\vec{x}) W_R(|\vec{x} - \vec{y}|) \quad (58)$$

the variance of the filtered density-contrast field is

$$\sigma^2_R = 4\pi \int \frac{k^2 dk}{(2\pi)^3} P(k) \hat{W}_R^2(k) \quad (59)$$

$\sigma_8$ is often used for normalising the power spectrum
modes entering the horizon (Hubble radius) while radiation dominates are relatively suppressed compared to larger modes which enter the horizon afterwards assuming that the fluctuation power entering the horizon should not depend on time, and working out the suppression for $k > k_0$ yields

$$P(k) \propto \begin{cases} 
  k & (k < k_0) \\
  k^{-3} & (k \gg k_0)
\end{cases} \quad (60)$$

this is the shape of the spectrum for cold dark matter (CDM); hot dark matter (HDM) cuts off the spectrum exponentially
Growth Suppression

\[ \delta \propto \Omega^2 \]

\[ \left( \frac{a_{\text{enter}}}{a_{\text{eq}}} \right)^2 \]

\[ \propto \Omega \]

\[ a_{\text{eq}} \]

\[ a_{\text{enter}} \]
The Cosmological Standard Model and its Parameters
The Observed Power Spectrum

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The Zel’dovich Approximation

- A kinematical treatment for following density evolution into the non-linear regime was invented by Zel’dovich; particle trajectories are approximated by

\[ \vec{r} = a \left[ \vec{x} + \frac{\vec{u}}{Hf(\Omega)} \right], \quad F_{ij} \equiv \frac{\partial r_i}{\partial x_j} \]  

(61)

- Important consequence is the probability distribution \( p(\lambda_1, \lambda_2, \lambda_3) \) for the eigenvalues of the deformation tensor \( F_{ij} \):

\[ p(\lambda_1, \lambda_2, \lambda_3) \propto |(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)| \]  

(62)

Probability for two eigenvalues of \( F_{ij} \) to be equal is zero, implying anisotropic collapse! (starting from Gaussian random field)
correct treatment of non-linear evolution requires numerical simulations; decompose the matter distribution into particles whose equations of motion are solved

non-linear evolution causes mode coupling: while modes of different wave lengths evolve independently during linear evolution, mode coupling in the non-linear evolution moves power from large to small scales as structures collapse

even if the original density perturbation field $\delta$ is Gaussian, it must develop non-Gaussianities during non-linear evolution

typical behaviour seen in numerical simulations shows the formation of “pancakes”, filaments and voids
Chapter VII

Structures in the Cosmic Microwave Background
Structures in the Cosmic Microwave Background

1. Simplified Theory of CMB Temperature Fluctuations
2. CMB Power Spectra and Cosmological Parameters
3. Foregrounds
Earth’s motion causes temperature dipole,

\[ T(\theta) = T_0 \left( 1 + \frac{v}{c} \cos \theta \right) + O\left( \frac{v^2}{c^2} \right) \]  (63)

\( \delta \gtrsim 1 \) today implies

\[ \delta(a_{\text{CMB}}) = \frac{\delta(a = 1)}{D_+(a_{\text{CMB}})} \gtrsim a_{\text{CMB}}^{-1} \approx 10^{-3} \]  (64)

and similar temperature fluctuations in the CMB

such fluctuations are not found; based on the assumption of weakly interacting dark matter, temperature fluctuations are expected to be \( \delta T/T \approx 10^{-5} \): detected by COBE in 1992
Simplified Theory: Sachs-Wolfe Effect, Acoustic Oscillations

- Perturbation equation for relative temperature fluctuation $\Theta \equiv \delta T / T_0$:

$$\ddot{\Theta} + \frac{c^2 k^2}{3} \dot{\Theta} - \frac{k^2}{3} \delta \dot{\Phi} - \frac{\delta \ddot{\Phi}}{c^2} = 0 \quad (65)$$

- For small $k$: Sachs-Wolfe-effect, $\dot{\Theta} \propto \delta \dot{\Phi} / c^2$

- Otherwise, oscillator equation for $\dot{\Theta} - \delta \dot{\Phi} / c^2 \equiv \dot{\theta}$; solution assuming $\dot{\theta} = 0$ at $t = 0$:

$$\hat{\theta}(t_{\text{rec}}) = \hat{\theta}(0) \cos \left[ \frac{ck}{\sqrt{3}} t_{\text{rec}} \right] \quad (66)$$

$c / \sqrt{3} t_{\text{rec}} \equiv r_s$: sound horizon; oscillations for $k > 2\pi / r_s$
damping occurs due to photon diffusion; diffusion scale:

\[ \lambda_D = \sqrt{N\lambda}, \quad \lambda = \frac{1}{n_e \sigma_T} \]  

(67)

number of collisions per unit time is \( dN = n_e \sigma_T c \, dt \); thus,

\[ \lambda_D^2 = \int_0^{t_{rec}} \frac{c \, dt}{n_e \sigma_T} \]  

(68)

structures smaller than the diffusion length are damped, hence damping sets in for wave numbers

\[ k > k_D = \frac{2\pi}{\lambda_D} \]  

(69)
CMB Spectra

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Thomson scattering is anisotropic; its differential cross section is

\[
\frac{d\sigma}{d\Omega} = \frac{3\sigma_T}{8\pi} |\vec{e}' \cdot \vec{e}|^2
\]  

(70)

\(\vec{e}'\) and \(\vec{e}\) are the unit vectors of the incoming and outgoing electric fields.

If infalling radiation has quadrupolar intensity anisotropy, scattered radiation is polarised because it has different intensities in its two orthogonal polarisation directions.

CMB is expected to be linearly polarised to some degree; the intensity of the polarised light should be of order 10% that of the unpolarised light, i.e. it should have an amplitude of order 10^{-6} K.
Polarisation

Quadrupole Anisotropy

\[ \epsilon', \epsilon' \]

Thomson Scattering

\[ \epsilon \]

Linear Polarization

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Instruments

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The Cosmological Standard Model and its Parameters
Measurements

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<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>CMB temperature</td>
<td>$2.726 \pm 0.002$ K</td>
</tr>
<tr>
<td>total density</td>
<td>$1.02 \pm 0.02$</td>
</tr>
<tr>
<td>matter density</td>
<td>$0.27 \pm 0.04$</td>
</tr>
<tr>
<td>baryon density</td>
<td>$0.044 \pm 0.004$</td>
</tr>
<tr>
<td>Hubble constant</td>
<td>$0.71^{+0.04}_{-0.03}$</td>
</tr>
<tr>
<td>baryon-to-photon ratio</td>
<td>$6.1^{+0.3}_{-0.2} \times 10^{-10}$</td>
</tr>
<tr>
<td>fluctuation amplitude</td>
<td>$0.84 \pm 0.04$</td>
</tr>
<tr>
<td>scalar spectral index</td>
<td>$0.93 \pm 0.03$</td>
</tr>
<tr>
<td>decoupling redshift</td>
<td>$1089 \pm 1$</td>
</tr>
<tr>
<td>age of the Universe</td>
<td>$13.7 \pm 0.2$ Gyr</td>
</tr>
<tr>
<td>age at decoupling</td>
<td>$379^{+8}_{-7}$ kyr</td>
</tr>
<tr>
<td>reionisation redshift (95% c.l.)</td>
<td>$20^{+10}_{-9}$</td>
</tr>
<tr>
<td>reionisation optical depth</td>
<td>$0.17 \pm 0.04$</td>
</tr>
</tbody>
</table>
Foregrounds

- CMB shines through the entire visible universe on its way to us
- microwave emission from our own Galaxy: warm dust in the plane of the Milky Way with a temperature near 20 K; synchrotron emission from electrons gyrating in the Galactic magnetic field; thermal bremsstrahlung from ionised hydrogen; line emission from molecules like CO
- hot plasma in galaxy clusters inverse-Compton scatters microwave background photons to higher energies: Sunyaev-Zel’dovich effect
- point source appearing in the microwave background, such as high-redshift galaxies, planets, asteroids, possibly comets in the Solar System, dust in the plane of the Solar System
Foreground Components

incoming, low-energy photon
outgoing, higher-energy photon
hot electrons in galaxy cluster
observer
Planck

The Cosmological Standard Model and its Parameters
Chapter VIII

Cosmological Inflation
Cosmological Inflation

1. Problems
2. Inflation
angular size of the particle horizon at recombination is

\[ \theta_{\text{rec}} = \frac{a_{\text{rec}} \Delta w(0, a_{\text{rec}})}{D_{\text{ang}}(0, z_{\text{rec}})} \approx \sqrt{\Omega_0 a_{\text{rec}}} \approx 1.7^\circ \sqrt{\Omega_0} \quad (71) \]

causal connection? horizon problem

evolution of flatness:

\[ |\Omega_{\text{total}} - 1| \propto \begin{cases} t & \text{radiation-dominated era} \\ t^{2/3} & \text{early matter-dominated era} \end{cases} \quad (72) \]

tiny deviations of \( \Omega_{\text{total}} \) from unity grow rapidly! flatness problem

where do structures originate from in the first place?
Horizon Problem
universe can be driven towards flatness by accelerated expansion, $\ddot{a} > 0$; this seems incompatible with gravity.

Friedmann’s equation allows accelerated expansion if

$$\rho c^2 + 3p < 0, \quad p < -\frac{\rho c^2}{3} \quad (73)$$

simple scalar field with Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \quad (74)$$

has negative pressure if

$$\dot{\phi}^2 < V(\phi) \quad (75)$$
Inflation: Slow Roll, Conditions

- successful inflation under slow-roll conditions:

\[
\epsilon \equiv \frac{1}{24\pi G} \left( \frac{V'}{V} \right)^2 \ll 1, \quad \eta \equiv \frac{1}{8\pi G} \left( \frac{V''}{V} \right) \ll 1
\] (76)

- for solving the flatness problem, increase in the scale factor by a factor of approximately \(e^{60}\) is necessary; this would at the same time solve the horizon (or causality) problem

- it is assumed that the inflaton field can decay through some coupling to “ordinary” matter: *reheating*
Slow Roll

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Inflation: Structure Formation

- Inflaton field must have undergone vacuum fluctuations.
- Once inflation sets in, they are quickly driven outside of the horizon, where they “freeze in” because they lack causal contact.
- The (primordial) density power spectrum predicted by inflation is
  \[ P_i(k) \propto k^n, \quad n \lesssim 1 \]  (77)
- Density fluctuations are expected to be Gaussian because of the central limit theorem.
- Inflation provides a possibility for solving the horizon and flatness problems and provides a natural explanation for the origin of structures in the universe.
Causality and Structure Formation

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Chapter IX

Dark Energy
1. Expansion of the Universe
2. Modified Equation of State
3. Effects on Cosmology
CMB measurements: universe is spatially flat, i.e. its total energy density equals the critical density.

Dark and baryonic matter density, dark and baryonic contributes approximately 30% to the total energy density; abundance of light elements requires the baryon density to be much lower.

Supernovae of type Ia: fixed amount of “explosives” blows up; they form a class of “standard candles”.

Comparing absolute luminosity and apparent brightness, luminosity distance as a function of redshift can be inferred.

High-redshift supernovae show transition from deceleration to acceleration near $z \sim 1$. 
Type-Ia Supernovae

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cosmological constant is physically dissatisfactory; as for inflation, assume scalar field ("cosmon", "quintessence") with negative pressure,

\[ p = w \rho c^2, \quad w < -\frac{1}{3} \]  

if \( w \) is constant,

\[ \rho_Q = \rho_{Q0} a^{-3(1+w)} \]  

Friedmann equation becomes

\[ H^2(a) = H_0^2 \left[ \Omega_{m0} a^{-3} + (1 - \Omega_{m0} - \Omega_{Q0}) a^{-2} + \Omega_{Q0} a^{-3(1+w)} \right] \]
Equation of State

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cosmic expansion during nucleosynthesis is tightly constrained by light-element abundances

effects on the CMB: width of the recombination shell, amount of Silk damping

modified angular-diameter and luminosity distances affect supernovae of type Ia, apparent size of CMB fluctuations, cosmic volume, overall geometry of the universe, and gravitational lensing

the growth factor is modified; structures form earlier in quintessence compared to cosmological-constant models

dark-matter haloes tend to be denser, which may have strong effects on their appearance
Growth Factor, Distances

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Cosmic Concordance

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### Baryon density $\Omega_b h^2$

- $\Omega_b h^2 = 0.0223 \pm 0.0008$
- $\eta_{10} = 6.1^{+0.3}_{-0.2}$

### Abundance Ratios

- $^4\text{He}/(\text{H}^4\text{He})$
- $^3\text{He}/\text{H}$
- $^7\text{Li}/\text{H}$

### Baryon-to-photon ratio $\eta_{10}$

- $10^{-3}$
- $10^{-4}$
- $10^{-5}$
- $10^{-9}$
- $10^{-10}$

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